Functional Analysis

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Lecture 13

Banach's open mapping theorem

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Def. $f : X \to Y$ **open** $\overset{def}{\iff} \underset{U \subseteq X \text{ open}}{\forall} f(U)$ open in Y

Rem. A bijective map $f : X \to Y$ is open $\iff$ the inverse $f^{-1} : Y \to X$ is continuous.

A continuous bijection $f : X \to Y$ is a homeomorphism $\iff$ it is an open map.

Let $K_X := \{x \in X : \|x\| < 1\}$ be the unit ball in the normed space X. The ball with center $x_0 \in X$ and radius $r > 0$ can be written as $x_0 + rK_X = \{x_0 + ry : y \in K_x\}$.

Lem. Let $T : X \to Y$ be a linear operator and let K_X and K_Y be unit balls in normed spaces X and Y.

$$T \text{ is an open map} \iff \underset{r>0}{\exists}\ rK_Y \subseteq T(K_X).$$

Moreover if T is open, it has to be surjective.

$$T \text{ is an open map} \iff \underset{r>0}{\exists}\ rK_Y \subseteq T(K_X).$$

Proof:

"$\Longrightarrow$" If T is open, then $T(K_X)$ is an open set. Since $0 \in T(K_X)$, there is $r > 0$ such that $rK_Y \subseteq T(K_X)$. Moreover

$$Y = \bigcup_{n=1}^{\infty} nrK_Y \subseteq \bigcup_{n=1}^{\infty} nT(K_X) = T(\bigcup_{n=1}^{\infty} nK_X) = T(X).$$

Whence $T(X) = Y$.

"$\Longleftarrow$" Assume that $rK_Y \subseteq T(K_X)$ for some $r > 0$. Take open $U \subseteq X$. Let $y \in T(U)$ and let $x \in U$ be such that $Tx = y$. Since U is open, there is $\delta > 0$ such that $x + \delta K_X \subseteq U$. Note that

$$y + \delta r K_Y \subseteq y + \delta T(K_X) = Tx + \delta T(K_X) = T(x + \delta K_X) \subseteq T(U).$$

Hence every point in $T(U)$ is in the interior of $T(U)$. That is $T(U)$ is an open set. ∎

Banach's open operator theorem

Let $T \in B(X, Y)$, where X and Y Banach spaces.

$$T \text{ is surjective} \iff T \text{ is open.}$$

Dowód: "$\Longleftarrow$" It follows from **Lem**.
"$\Longrightarrow$"Assume that T is a surjection. Then

$$Y = T(X) = T(\bigcup_{n=1}^{\infty} nK_X) = \bigcup_{n=1}^{\infty} T(nK_X).$$

By **Baire's theorem** (as Y is complete) there is $n \in \mathbb{N}$ such that

$$\mathrm{Int}(\overline{T(nK_X)}) \neq \emptyset.$$

Hence there is $y_0 \in Y$ and $\varepsilon > 0$ such that $y_0 + \varepsilon K_Y \subseteq \overline{T(nK_X)}$.
Since $T(X) = Y$, there is $x_0 \in X$ such that $Tx_0 = y_0$. Whence

$$\varepsilon K_Y \subseteq \overline{T(nK_X)} - y_0 = \overline{T(nK_X)} - T(x_0) = \overline{T(nK_X - x_0)}$$
$$\subseteq \overline{T((n + \|x_0\|)K_X)} = (n + \|x_0\|)\overline{T(K_X)}.$$

Dividing by $n + \|x_0\|$ and putting $r := \frac{\varepsilon}{n+\|x_0\|}$ we get

$$rK_Y \subseteq \overline{T(K_X)}. \tag{1}$$

Up to a closure it is a condition from **Lem**. To "get rid of the closure" we show that

$$\overline{T(K_X)} \subseteq T(2K_X). \tag{2}$$

Let $y \in \overline{T(K_X)}$. There is $x_1 \in K_X$ such that $\|y - Tx_1\| < \frac{r}{2}$. Hence

$$y - Tx_1 \in \tfrac{r}{2}K_Y \overset{(1)}{\subseteq} \tfrac{1}{2}\overline{T(K_X)} = \overline{T\left(\tfrac{1}{2}K_X\right)}.$$

Applying the same argument to $y - Tx_1 \in \overline{T\left(\frac{1}{2}K_X\right)}$ we may find $x_2 \in \frac{1}{2}K_X$ such that $\|(y - Tx_1) - Tx_2\| < \frac{r}{4}$ and therefore

$$y - T(x_1 + x_2) = (y - Tx_1) - Tx_2 \in \tfrac{r}{4}K_Y \subseteq \overline{T\left(\tfrac{1}{4}K_X\right)}.$$

Continuing in this manner we get a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ where

$$x_n \in \frac{1}{2^{n-1}}K_X \quad \text{and} \quad y - T(x_1 + ... + x_n) \in \frac{r}{2^n}K_Y.$$

This second relation implies that $T(x_1 + ... + x_n) \to y$ in Y.

While the firs relation (by completeness of X) guarantees that the series $\sum_{n=1}^{\infty} x_n$ converges in X, as it is absolutely convergent:

$$\left\| \sum_{n=1}^{\infty} x_n \right\| \leqslant \sum_{n=1}^{\infty} \|x_n\| < \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2.$$

In particular, $\sum_{n=1}^{\infty} x_n \in 2K_X$. Using continuity (boundedness) of the operator T we get

$$T\left(\sum_{n=1}^{\infty} x_n\right) = T\left(\lim_{n\to\infty} x_1 + \ldots + x_n\right) = \lim_{n\to\infty} T(x_1 + \ldots + x_n) = y,$$

Thus $y \in T(2K_X)$. This proves the inclusion in (2).

Together with the inclusion (1) this gives $rK_Y \subseteq T(2K_X)$ or equivalently $\frac{r}{2}K_Y \subseteq T(K_X)$. Hence T is an open map by **Lem**. ■

Cor1. $\left(\begin{array}{l} T \in B(X,Y) \text{ and } T \text{ bijection} \\ X, Y \text{ Banach spaces} \end{array}\right) \implies T^{-1} \in B(Y,X)$

Proof: Since T is surjective, it is open by the Open Mapping Theorem. Hence for every open $U \subseteq X$ the set $(T^{-1})^{-1}(U) = T(U)$ is open in Y. Thus the operator T^{-1} is continuous, and therefore bounded. ■

Cor2. Every two comparable complete norms on X are equivalent.

Proof: Recall that a norm $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$ if

$$\exists_{c_1>0}\ \forall_{x\in X}\ \|x\|_1 \leqslant c_1\|x\|_2,$$

that is the identity operator $id : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ is bounded. Since id is bijective, its inverse $id : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is bounded by **Cor1**. That is

$$\exists_{c_2>0}\ \forall_{x\in X}\ \|x\|_2 \leqslant c_2\|x\|_1,$$

which means that the norm $\|\cdot\|_2$ is weaker than $\|\cdot\|_1$. Hence the two norms are equivalent. ■

Def. The **graph of a function** $f : X \to Y$ is the set

$$\Gamma(f) := \{(x, f(x)) : x \in X\} \subseteq X \times Y.$$

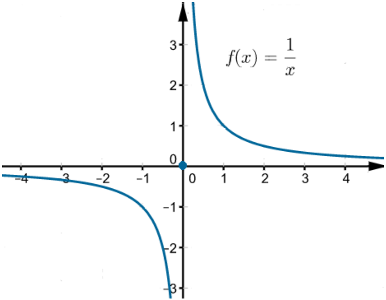
Lem. The graph of a continuous function is closed:

$$\left(\begin{array}{l} f : X \to Y \text{ continuous} \\ X, Y \text{ metric spaces} \end{array} \right) \implies \Gamma(f) \text{ closed in } X \times Y.$$

Proof: If $(x_0, y_0) \in \overline{\Gamma(f)}$, there is a sequence $(x_n, y_n) \in \Gamma(f)$ such that $(x_n, y_n) \to (x_0, y_0)$. By continuity

$$f(x_0) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} y_n = y_0$$

Hence $(x_0, y_0) \in \Gamma(f)$. ■



Ex. The converse implication in **Lem** does not hold. Let $X = Y = \mathbb{R}$ and

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Thm. (Closed graph theorem)

A linear operator $T : X \to Y$ between two Banach spaces is continuous (bounded) $\Longleftrightarrow$ the graph of T is a closed set.

Proof: We only need to show '$\Longleftarrow$'. Note that

1) $X \times Y$ is a Banach space with $\|(x, y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$
2) $\Gamma(T)$ is a closed linear subspace of $X \times Y$.

Hence $\Gamma(T)$ is a Banach space with the norm $\|\cdot\|_{X\times Y}$. Projections

$$P_1 : \Gamma(T) \to X, \quad \text{where} \quad P_1(x, Tx) = x,$$

$$P_2 : \Gamma(T) \to Y, \quad \text{where} \quad P_2(x, Tx) = Tx,$$

are linear and bounded ($\|P_1\| \leqslant 1$, $\|P_2\| \leqslant 1$). In addition P_1 is invertible. Hence its inverse

$$P_1^{-1} : X \to \Gamma(T), \quad \text{where} \quad P_1^{-1}(x) = (x, Tx)$$

is bounded by **Cor1**. Hence the operator

$$T = P_2 \circ P_1^{-1}$$

is bounded as a composition bounded operators. ■